Coloring intersection graphs of x-monotone curves in the plane

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Abstract

A class of graphs $\mathcal{G}$ is χ-*bounded* if the chromatic number of the graphs in $\mathcal{G}$ is bounded by some function of their clique number. We show that the class of intersection graphs of simple x-monotone curves in the plane intersecting a vertical line is χ-bounded. As a corollary, the class of intersection graphs of rays in the plane is χ-bounded.

1 Introduction

For a graph G, The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors required to color the vertices of G such that any two adjacent vertices have distinct colors. The *clique number* of G, denoted by $\omega(G)$, is the size of the largest clique in G. We say that a class of graphs $\mathcal{G}$ is χ-*bounded* if there exists a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that every $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. Although there are triangle-free graphs with arbitrarily large chromatic number [4, 21], it has been shown that certain graph classes arising from geometry are χ-bounded.

Given a collection of objects $\mathcal{F}$ in the plane, the *intersection graph* $G(\mathcal{F})$ has vertex set $\mathcal{F}$ and two objects are adjacent if and only if they have a nonempty intersection. For simplicity, we will shorten $\chi(G(\mathcal{F})) = \chi(\mathcal{F})$ and $\omega(G(\mathcal{F})) = \omega(\mathcal{F})$. The study of the chromatic number of intersection graphs of objects in the plane was stimulated by the seminal papers of Asplund and Grünbaum [2] and Gyárfás and Lehel [8, 9]. Asplund and Grünbaum showed that if $\mathcal{F}$ is a family of axis parallel rectangles in the plane, then $\chi(\mathcal{F}) \leq 4\omega(\mathcal{F})^2$. Gyárfás and Lehel [8, 9] showed that if $\mathcal{F}$ is a family of chords in a circle, then $\chi(\mathcal{F}) \leq 2^{\omega(\mathcal{F})}\omega(\mathcal{F})^3$. Over the past 50 years, this topic has received a large amount of attention due to its application in VLSI design [10], map labeling [1], graph drawing [5, 19], and elsewhere. For more results on the chromatic number of intersection graphs of objects in the plane and in higher dimensions, see [3, 5, 9, 11, 12, 13, 14, 16, 17, 15].

In this paper, we study the chromatic number of intersection graphs of x-monotone curves in the plane. We say a family of curves $\mathcal{F}$ is *simple*, if every pair of curves intersect at most once. Our main theorem is the following.

Theorem 1.1. *The class of intersection graphs of simple x-monotone curves in the plane intersecting a vertical line is χ-bounded.*

In [16, 17], McGuinness proved a similar statement for triangle-free intersection graphs of curves in the plane. As an immediate corollary of Theorem 1.1, we have the following.

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Corollary 1.2. *The class of intersection graphs of rays in the plane is χ-bounded.*

Let us remark that the following problem is still open.

Problem 1.3 ([5, 13, 14])**.** *Is the class of intersection graphs of segments in the plane χ-bounded?*

By applying partitioning [7] and divide and conquer [18] arguments, Theorem 1.1 also implies the following results. Since these arguments are fairly standard, we omit their proofs.

Theorem 1.4. *Let $\mathcal{S}$ be a family of segments in the plane, such that no k members pairwise cross. If the ratio of the longest segment to the shortest segment is bounded by r, then $\chi(\mathcal{S}) \leq c_{k,r}$ where $c_{k,r}$ depends only on k and r.*

Theorem 1.5. *Let $\mathcal{F}$ be a family of n simple x-monotone curves in the plane, such that no k members pairwise cross. Then $\chi(\mathcal{F}) \leq c_k \log n$, where c_k is a constant that depends only on k.*

This improves the previous known bound of $(\log n)^{15 \log k}$ due to Fox and Pach [5]. We note that the Fox and Pach bound holds without the simple condition.

Recall that a *topological graph* is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by non-self-intersecting arcs connecting the corresponding points. A topological graph is *simple* if every pair of its edges intersect at most once. Since every n-vertex planar graph has at most $3n-6$ edges, Theorem 1.1 gives a new proof of the following result due to Valtr.

Theorem 1.6 ([20])**.** *Let $G = (V, E)$ be an n-vertex simple topological graph with edges drawn as x-monotone curves. If there are no k pairwise crossing edges in G, then $|E(G)| \leq c_k n \log n$, where c_k is a constant that depends only on k.*

We note that Suk recently showed that Theorem 1.6 holds without the simple condition [19].

2 Definitions and notation

A family $\mathcal{F}$ of x-monotone curves in the plane is called a *left-flag* (*right-flag*) family, if the left (right) endpoint of each of its members lie on the y-axis. Hence in order to prove Theorem 1.1, it suffices to prove the following theorem on families of x-monotone curves that form a left-flag.

Theorem 2.1. *Let $\mathcal{F}$ be a family of simple x-monotone curves that form a left-flag. If $\chi(\mathcal{F}) > 2^{(5^{k+1}-121)/4}$, then $\mathcal{F}$ contains k pairwise crossing members.*

The rest of this paper is devoted to proving Theorem 2.1. Given a family $\mathcal{F} = \{C_1, C_2, ..., C_n\}$ of n simple x-monotone curves that form a left-flag, we can assume that no two curves share a point on the y-axis and the curves are ordered from bottom to top. We let $G(\mathcal{F})$ be the intersection graph of $\mathcal{F}$ such that vertex $i \in V(G(\mathcal{F}))$ corresponds to the curve C_i. We can assume that $G(\mathcal{F})$ is connected. For a given curve $C_i \in \mathcal{F}$, we say that C_i is at *distance* d from $C_1 \in \mathcal{F}$, if the shortest path from vertex 1 to i in $G(\mathcal{F})$ has length d. We call the sequence of curves $C_{i_1}, C_{i_2}, ..., C_{i_p}$ a *path* if the corresponding vertices in $G(\mathcal{F})$ form a path, that is, the curve C_{i_j} intersects $C_{i_{j+1}}$ for $j = 1, 2, ..., p-1$. For $i < j$, we say that C_i *lies below* C_j, and C_j *lies above* C_i. We denote $x(C_i)$ to be the x-coordinate of the right endpoint of the curve C_i. Given a subset of curves $\mathcal{K} \subset \mathcal{F}$, we denote

$$x(\mathcal{K}) = \min_{C \in \mathcal{K}}(x(C)).$$

For any subset $I \subset \mathbb{R}$, we let $\mathcal{F}(I) = \{C_i \in \mathcal{F} : i \in I\}$. If I is an interval, we will shorten $\mathcal{F}((i,j))$ to $\mathcal{F}(i,j)$, $\mathcal{F}([i,j])$ to $\mathcal{F}[i,j]$, $\mathcal{F}((i,j])$ to $\mathcal{F}(i,j]$, and $\mathcal{F}([i,j))$ to $\mathcal{F}[i,j)$.

For $\alpha \geq 0$, a finite sequence $\{r_i\}_{i=0}^m$ of $\mathcal{F}$ is called an *α-sequence* if for $r_0 = \min\{i : C_i \in \mathcal{F}\}$ and $r_m = \max\{i : C_i \in \mathcal{F}\}$, the subsets $\mathcal{F}[r_0, r_1], \mathcal{F}(r_1, r_2], ..., \mathcal{F}(r_{m-1}, r_m]$ satisfy

$$\chi(\mathcal{F}[r_0, r_1]) = \chi(\mathcal{F}(r_1, r_2]) = \cdots = \chi(\mathcal{F}(r_{m-2}, r_{m-1}]) = \alpha$$

and

$$\chi(\mathcal{F}(r_{m-1}, r_m]) \leq \alpha.$$

3 Combinatorial coloring lemmas

We will make use of the following lemmas. The first lemma is on ordered graphs $G = ([n], E)$, whose proof can be found in [15]. For sake of completeness, we shall add the proof. Just as before, for any interval $I \subset \mathbb{R}$, we denote $G(I) \subset G$ to be the subgraph induced by vertices $V(G) \cap I$.

Lemma 3.1. *Given a graph $G = ([n], E)$, let $a, b \geq 0$ and suppose that $\chi(G) > 2^{a+b+1}$. Then there exists an induced subgraph $H \subset G$ where $\chi(H) > 2^a$, and for all $uv \in E(H)$ we have $\chi(G(u,v)) \geq 2^b$.*

Proof. Let $\{r_i\}_{i=0}^m$ be a 2^b-sequence of $V(G)$. Then for $r_0 = 1$ and $r_m = n$, we have subgraphs $G[r_0, r_1], G(r_1, r_2], ..., G(r_{m-1}, r_m]$, such that

$$\chi(G[r_0, r_1]) = \chi(G(r_1, r_2]) = \cdots = \chi(G(r_{m-2}, r_{m-1}]) = 2^b$$

and

$$\chi(G(r_{m-1}, r_m]) \leq 2^b.$$

For each of these subgraphs, we will properly color its vertices with colors, say, $1, 2, ..., 2^b$. Since $\chi(G) > 2^{a+b+1}$, there exists a color class for which the vertices of this color induces a subgraph with chromatic number at least 2^{a+1}. Let G' be such a subgraph, and we define subgraphs $H_1, H_2 \subset G'$ such that

$$H_1 = G'[0, r_1] \cup G'(r_2, r_3] \cup \cdots \qquad \text{and} \qquad H_2 = G'(r_1, r_2] \cup G'(r_3, r_4] \cup \cdots$$

Since $V(H_1) \cup V(H_2) = V(G')$, either $\chi(H_1) > 2^a$ or $\chi(H_2) > 2^a$. Without loss of generality, we can assume $\chi(H_1) > 2^a$ holds, and set $H = H_1$. Now for any $uv \in E(H)$, there exists integers i, j such that for $0 \leq i < j$, we have $u \in V(G'(r_{2i}, r_{2i+1}])$ and $v \in V(G'(r_{2j}, r_{2j+1}])$. This implies $G(r_{2i+1}, r_{2i+2}] \subset G(u,v)$ and

$$\chi(G(u,v)) \geq \chi(G(r_{2i+1}, r_{2i+2}]) = 2^b.$$

This completes the proof of the lemma. $\square$

Recall that the distance between two vertices $u, v \in V(G)$ in a graph G, is the length of the shortest path from u to v.

Lemma 3.2. *Let G be a graph and let $v \in V(G)$. Suppose $G^0, G^1, G^2, \dots$ are the subgraphs induced by vertices at distance $0, 1, 2, \dots$ respectively from v. Then for some d, $\chi(G_d) \geq \chi(G)/2$.*

Proof. For $0 \leq i < j$, if $|i-j| > 1$, then no vertex in G^i is adjacent to a vertex in G^j. By Pigeonhole, the statement follows.

$\square$

4 Using the induction hypothesis

The proof of Theorem 2.1 will be given in Section 5, and is done by induction on k. In particular, we will give a recursive construction of a function λ_k, such that if $\mathcal{F}$ is a family of simple x-monotone curves that form a left-flag with $\chi(\mathcal{F}) > 2^{\lambda_k}$, then $\mathcal{F}$ contains k pairwise crossing members. Note that λ_k will satisfy $\lambda_k > \log(2k)$ for all k. In the following two subsections, we shall assume that such a function exists (i.e. by the induction hypothesis) and prove several lemmas.

4.1 Key lemma

Let $\mathcal{F} = \{C_1, C_2, \dots, C_n\}$ be a family of n simple x-monotone curves that form a left-flag, such that no $k+1$ members pairwise cross. Suppose that curves C_a and C_b intersect, for $a < b$. Let

$$\begin{aligned}
\mathcal{I}_a &= \{C_i \in \mathcal{F}(a,b) : C_i \text{ intersects } C_a\}, \\
\mathcal{D}_a &= \{C_i \in \mathcal{F}(a,b) : C_i \text{ does not intersect } C_a\}.
\end{aligned}$$

We define $\mathcal{I}_b$ and $\mathcal{D}_b$ similarly and set $\mathcal{D}_{ab} = \mathcal{D}_a \cap \mathcal{D}_b \subset \mathcal{F}(a,b)$. Now we define three subsets of $\mathcal{D}_{ab}$ as follows:

$$\begin{aligned}
\mathcal{D}_{ab}^a &= \{C_i \in \mathcal{D}_{ab} : \exists C_j \in \mathcal{I}_a \text{ that intersects } C_i\}, \\
\mathcal{D}_{ab}^b &= \{C_i \in \mathcal{D}_{ab} : \exists C_j \in \mathcal{I}_b \text{ that intersects } C_i\}, \\
\mathcal{D} &= \mathcal{D}_{ab} \setminus (\mathcal{D}_{ab}^a \cup \mathcal{D}_{ab}^b).
\end{aligned}$$

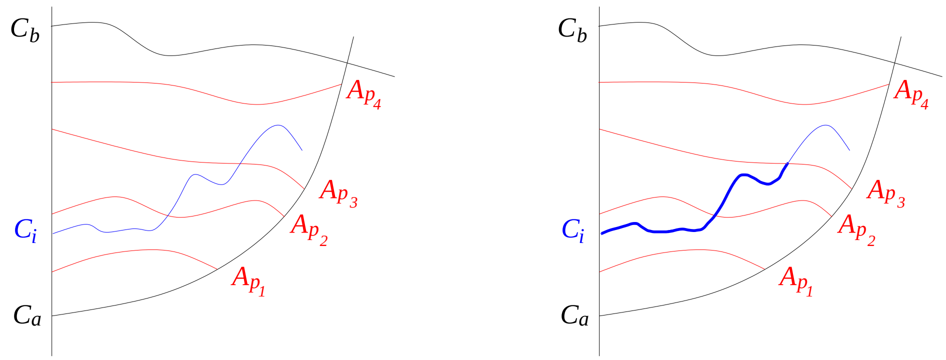
We now prove the following key lemma.

Lemma 4.1. $\chi(\mathcal{D}_{ab}^a \cup \mathcal{D}_{ab}^b) \leq k \cdot 2^{2\lambda_k + 102}$. *In other words,* $\chi(\mathcal{D}) \geq \chi(\mathcal{F}(a,b)) - 2^{\lambda_k+1} - k \cdot 2^{2\lambda_k+102}$.

Proof. Without loss of generality, we can assume that

$$\chi(\mathcal{D}_{ab}^a) \geq \frac{\chi(\mathcal{D}_{ab}^a \cup \mathcal{D}_{ab}^b)}{2}, \tag{1}$$

since otherwise a similar argument will follow if $\chi(\mathcal{D}_{ab}^b) \geq \chi(\mathcal{D}_{ab}^a \cup \mathcal{D}_{ab}^b)/2$. Let $\mathcal{D}_{ab}^a = \{C_{r_1}, C_{r_2}, \dots, C_{r_{m_1}}\}$ where $a < r_1 < r_2 < \dots < r_{m_1} < b$. For each curve $C_i \in \mathcal{I}_a$, we define A_i to be the arc along the



(a) In this example, $l(i) = 2$ and $u(i) = 3$.

(b) L_i drawn thick.

Figure 1: Curves in $\mathcal{S}_t^1$.

curve C_i, from the left endpoint of C_i to the intersection point $C_i \cap C_a$. Set $\mathcal{A} = \{A_i : \mathcal{C}_i \in \mathcal{I}_a\}$. Notice that the intersection graph of $\mathcal{A}$ is a perfect graph (see [6]). Since there are no $k+1$ pairwise crossing arcs in $\mathcal{A}$, we can decompose the members in $\mathcal{A}$ into k parts $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_k$, such that the arcs in $\mathcal{A}_i$ are pairwise disjoint. Then for $i = 1, 2, ..., k$, we define

$$\mathcal{S}_i = \{C_j \in \mathcal{D}_{ab}^a : C_j \text{ intersects an arc from } \mathcal{A}_i\}.$$

Since $\mathcal{D}_{ab}^a = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k$, there exists a $t \in \{1, 2, ..., k\}$ such that

$$\chi(\mathcal{S}_t) \geq \frac{\chi(\mathcal{D}_{ab}^a)}{k}. \tag{2}$$

Therefore, let $\mathcal{A}_t = \{A_{p_1}, A_{p_2}, ..., A_{p_{m_2}}\}$. Notice that each curve $C_i \in S_t$ intersects the members in $\mathcal{A}_t$ that lies either above or below C_i (but not both since $\mathcal{F}$ is simple). Moreover, C_i intersects the members in $\mathcal{A}_t$ in either increasing or decreasing order. Let $\mathcal{S}_t^1$ ($\mathcal{S}_t^2$) be the curves in $\mathcal{S}_t$ that intersects a member in $\mathcal{A}_t$ that lies above (below) it. Again, without loss of generality we will assume that

$$\chi(\mathcal{S}_t^1) \geq \frac{\chi(\mathcal{S}_t)}{2}, \tag{3}$$

since a symmetric argument will hold if $\chi(\mathcal{S}_t^2) \geq \chi(\mathcal{S}_t)/2$. For each curve $C_i \in \mathcal{S}_t^1$, we define

$$\begin{aligned} u(i) &= max\{j : \text{arc } A_{p_j} \in \mathcal{A}_t \text{ intersects } C_i\}, \\ l(i) &= min\{j : \text{arc } A_{p_j} \in \mathcal{A}_t \text{ intersects } C_i\}. \end{aligned}$$

See Figure 1(a) for a small example. Then for each curve $C_i \in \mathcal{S}_t^1$, we define the curve L_i to be the arc along C_i, joining the left endpoint of C_i and the point $C_i \cap A_{p_{u(i)}}$. See Figure 1(b). Now we set

$$\mathcal{L} = \{L_i : C_i \in \mathcal{S}_t^1\}.$$

Notice that $\mathcal{L}$ does not contain k pairwise crossing members. Indeed, otherwise these k curves $\mathcal{K} \subset \mathcal{L}$ would all intersect A_{p_i} where

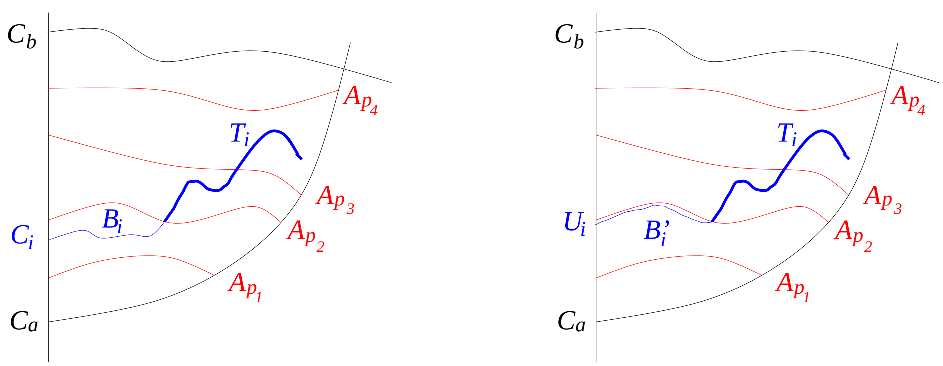
$$i = \min_{C_j \in \mathcal{K}} u(j),$$

creating $k+1$ pairwise crossing curves in $\mathcal{F}$. Therefore we can decompose $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \cdots \cup \mathcal{L}_w$ into w parts, such that $w \leq 2^{\lambda_k}$, and the set of curves in $\mathcal{L}_i$ are pairwise disjoint for $i = 1, 2, ..., w$. Let $\mathcal{H}_i \subset \mathcal{F}$ be the set of (original) curves corresponding to the (modified) curves in $\mathcal{L}_i$. Then there exists an $s \in \{1, 2, ..., w\}$ such that

$$\chi(\mathcal{H}_s) \geq \frac{\chi(\mathcal{S}_t^1)}{2^{\lambda_k}}. \tag{4}$$

Now for each curve $C_i \in \mathcal{H}_s$, we will define the curve U_i as follows. Let T_i be the arc along C_i, joining the right endpoint of C_i and the point $C_i \cap A_{p_{l(i)}}$. We define B_i to be the arc along C_i, joining the left endpoint of C_i and the point $C_i \cap A_{p_{l(i)}}$. See Figure 2(a). Notice that for any two curves $C_i, C_j \in \mathcal{H}_s$, B_i and B_j are disjoint. We define $U_i = B_i' \cup T_i$, where B_i' is the arc obtained by pushing B_i upwards so that it is "very" close to the curve $A_{p_{l(i)}}$. See Figure 2(b). We do this for every curve $C_i \in \mathcal{H}_s$, to obtain the family $\mathcal{U} = \{U_i : C_i \in \mathcal{H}_s\}$, such that

1. $\mathcal{U}$ is a family of simple x-monotone curves that form a left-flag,
2. we do not create any new crossing pairs in $\mathcal{U}$ (we may lose some crossing pairs),
3. any curve U_j that crosses B_i', must cross $A_{p_{l(i)}}$.



(a) T_i drawn thick and B_i drawn normal along C_i.

(b) Curve $U_i = B_i' \cup T_i$.

Figure 2: Curves in $\mathcal{S}_t^1$.

Notice that $\mathcal{U}$ does not contain k pairwise crossing members. Indeed, otherwise these k curves $\mathcal{K} \subset \mathcal{U}$ would all cross A_{p_i} where

$$i = \max_{U_j \in \mathcal{K}} l(j),$$

creating $k+1$ pairwise crossing curves in $\mathcal{F}$. Therefore we can decompose $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \cdots \cup \mathcal{U}_z$ into z parts, such that $z \leq 2^{\lambda_k}$ and the curves in $\mathcal{U}_i$ are pairwise disjoint. Let $\mathcal{C}_i \subset \mathcal{F}$ be the set of (original) curves corresponding to the (modified) curves in $\mathcal{U}_i$. Then there exists an $h \in \{1, 2, ..., z\}$ such that

$$\chi(\mathcal{C}_h) \geq \frac{\chi(\mathcal{H}_s)}{2^{\lambda_k}}. \tag{5}$$

Now we make the following observation.

Observation 4.2. *There are no three pairwise crossing curves in $\mathcal{C}_h$.*

Proof. Suppose that the pair of curves $C_i, C_j \in \mathcal{C}_h$ intersect, for $i < j$. Then we must have $i < p_{u(i)} < j < p_{l(j)}$ and $A_{p_{u(i)+1}} = A_{p_{l(j)}}$. Basically the "top tip" of C_i must intersect the "bottom tip" of C_j. See Figure 3. It is impossible for 3 curves to pairwise cross satisfying these properties.

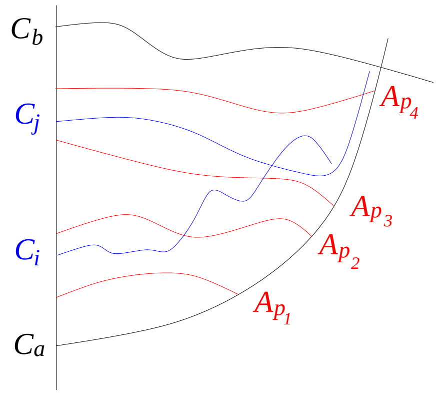


Figure 3: C_i and C_j crossing.

□

By a result of McGuinness [16], we know that

$$\chi(\mathcal{C}_h) \leq 2^{100}. \tag{6}$$

Therefore, by combining equations (1),(2),(3),(4),(5), and (6), we have

$$\chi(\mathcal{D}^a_{ab} \cup \mathcal{D}^b_{ab}) \leq k \cdot 2^{2\lambda_k + 102}.$$

Since both $\mathcal{I}_a$ and $\mathcal{I}_b$ doesn't contain k pairwise crossing members, we have $\chi(\mathcal{I}_a), \chi(\mathcal{I}_b) \leq 2^{\lambda_k}$. Therefore

$$\begin{aligned} \chi(\mathcal{D}) &\geq \chi(\mathcal{F}(a,b)) - \chi(\mathcal{I}_a) - \chi(\mathcal{I}_b) - \chi(\mathcal{D}^a_{ab} \cup \mathcal{D}^b_{ab}) \\ &\geq \chi(\mathcal{F}(a,b)) - 2^{\lambda_k + 1} - k \cdot 2^{2\lambda_k + 102}. \end{aligned}$$

□

Therefore, if there exists a curve $C_i \in \mathcal{F}$ that intersects C_a (or C_b) and a curve from $\mathcal{D}$, then $i < a$ or $i > b$.

4.2 Finding special configurations

In this section, we will show that if the chromatic number of $G(\mathcal{F})$ is sufficiently high, then certain subconfigurations must exist. We say that the set of curves $\{C_{i_1}, C_{i_2}, ..., C_{i_{k+1}}\}$ form a *type 1 configuration*, if

1. $i_1 < i_2 < \cdots < i_{k+1}$,
2. the set of k curves $\mathcal{K} = \{C_{i_1}, C_{i_2}, ..., C_{i_k}\}$ pairwise intersect,
3. $C_{i_{k+1}}$ does not intersect any of the curves in $\mathcal{K}$, and
4. $x(C_{i_{k+1}}) < x(\mathcal{K})$. See Figure 4(a).

Likewise, we say that the set of curves $\{C_{i_1}, C_{i_2}, ..., C_{i_{k+1}}\}$ form a *type 2 configuration*, if

1. $i_1 < i_2 < \cdots < i_{k+1}$,
2. the set of k curves $\mathcal{K} = \{C_{i_2}, C_{i_3}, ..., C_{i_{k+1}}\}$ pairwise intersect,
3. C_{i_1} does not intersect any of the curves in $\mathcal{K}$, and
4. $x(C_{i_1}) < x(\mathcal{K})$. See Figure 4(b).

We say that the set of curves $\{C_{i_1}, C_{i_2}, ..., C_{i_{2k+1}}\}$ form a *type 3 configuration*, if

1. $i_1 < i_2 < \cdots < i_{2k+1}$,
2. the set of k curves $\mathcal{K}_1 = \{C_{i_1}, ..., C_{i_k}\}$ pairwise intersect,
3. the set of k curves $\mathcal{K}_2 = \{C_{i_{k+2}}, C_{i_{k+3}}, ..., C_{i_{2k+1}}\}$ pairwise intersect,
4. $C_{i_{k+1}}$ does not intersect any of the curves in $\mathcal{K}_1 \cup \mathcal{K}_2$, and
5. $x(C_{i_{k+1}}) \leq x(\mathcal{K}_1 \cup \mathcal{K}_2)$. See Figure 4(c).

Note that in a type 3 configuration, a curve in $\mathcal{K}_1$ may or may not intersect a curve in $\mathcal{K}_2$. The goal of this subsection will be to show that if $G(\mathcal{F})$ has large chromatic number, then it must contain a type 3 configuration. We start by proving several lemmas.

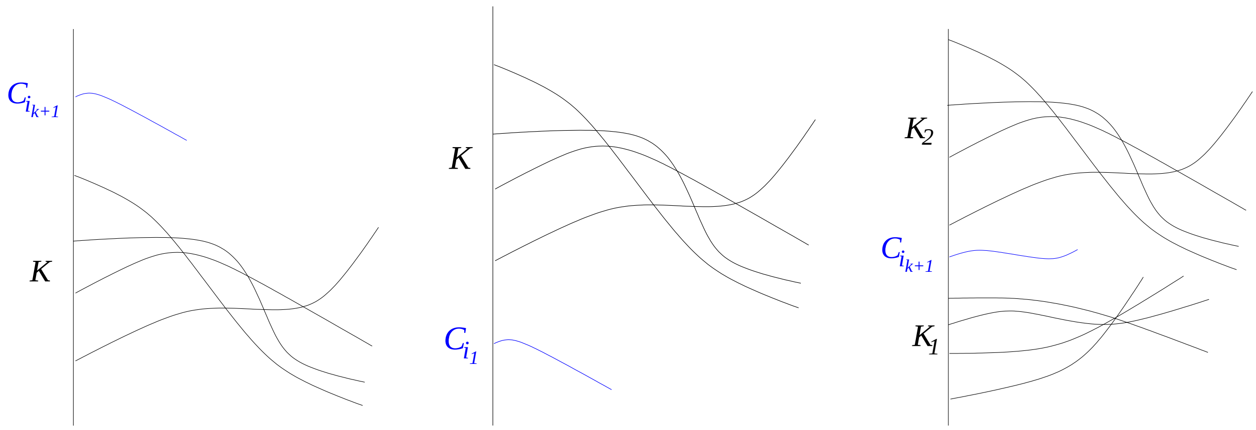
Lemma 4.3. *Let $\mathcal{F} = \{C_1, C_2, ..., C_n\}$ be a family of n x-monotone curves that form a left-flag. Suppose the set of curves $\mathcal{K} = \{C_{i_1}, C_{i_2}, ..., C_{i_m}\}$ are pairwise intersecting with $i_1 < i_2 < \cdots < i_m$. If there exits a curve C_j such that C_j is disjoint to all members in $\mathcal{K}$ and $i_1 < j < i_m$, then*

$$x(C_j) \leq x(\mathcal{K}).$$

Proof. Suppose that $x(C_{i_t}) < x(C_j)$ for some t. Without loss of generality, we can assume $i_1 < j < i_t$. Since C_{i_1} and C_{i_t} cross and are x-monotone, this implies that either C_{i_1} or C_{i_t} intersects C_j and therefore we have a contradiction. A symmetric argument holds if $i_t < j < i_m$.

□

Lemma 4.4. *Let $\mathcal{F} = \{C_1, C_2, ..., C_n\}$ be a family of n x-monotone curves that form a left-flag. Then for any set of t curves $C_{i_1}, C_{i_2}, ..., C_{i_t} \in \mathcal{F}$ where $t \leq 2k$, if $\chi(\mathcal{F}) > 2^{\beta}$, then either*



(a) Type 1 configuration. (b) Type 2 configuration. (c) Type 3 configuration.

Figure 4: Special configurations.

1. *$\mathcal{F}$ contains $k+1$ pairwise crossing members, or*
2. *there exists a subset $\mathcal{H} \subset \mathcal{F} \setminus \{C_{i_1}, C_{i_2}, ..., C_{i_t}\}$, such that each curve $C_j \in \mathcal{H}$ is disjoint to all members in $\{C_{i_1}, C_{i_2} ..., C_{i_t}\}$, and $\chi(\mathcal{H}) > 2^{\beta} - 2^{2\lambda_k}$.*

Proof. For each $j \in \{1, 2, ..., t\}$, let $\mathcal{H}_j \subset \mathcal{F}$ be the subset of curves that intersect C_{i_j}. If $\chi(\mathcal{H}_j) > 2^{\lambda_k}$ for some $j \in \{1, 2, ..., t\}$, then $\mathcal{F}$ contains $k+1$ pairwise crossing members. Therefore, we can assume that $\chi(\mathcal{H}_j) \leq 2^{\lambda_k}$ for all $1 \leq j \leq t$. Now let $\mathcal{H} \subset \mathcal{F}$ be the subset of curves defined by

$$\mathcal{H} = \mathcal{F} \setminus (\mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_t).$$

Since $\chi(\mathcal{F}) > 2^{\beta}$, we have

$$\chi(\mathcal{H}) > 2^{\beta} - t2^{\lambda_k} \geq 2^{\beta} - 2^{2\lambda_k},$$

where the last inequality follows from the fact that $\log t < \log 2k < \lambda_k$.

□

Lemma 4.5. *Let $\mathcal{F} = \{C_1, C_2, ..., C_n\}$ be a family of n x-monotone curves that form a left-flag. If $\chi(\mathcal{F}) \geq 2^{4\lambda_k + 107}$, then either*

1. *$\mathcal{F}$ contains $k+1$ pairwise crossing members, or*
2. *$\mathcal{F}$ contains a type 1 configuration, or*
3. *$\mathcal{F}$ contains a type 2 configuration.*

Proof. Assume that $\mathcal{F}$ does not contain $k+1$ pairwise crossing members. By Lemma 3.2, for some $d \geq 2$, the subset of curves $\mathcal{F}^d$ at distance d from curve C_1 satisfies

$$\chi(\mathcal{F}^d) \geq \frac{\chi(\mathcal{F})}{2} \geq 2^{4\lambda_k + 106}.$$

By Lemma 3.1, there exists a subset $\mathcal{H}_1 \subset \mathcal{F}^d$ such that $\chi(\mathcal{H}_1) > 2$, and for every pair of curves $C_a, C_b \in \mathcal{H}_1$ that intersect, $\mathcal{F}^d(a,b) \geq 2^{4\lambda_k+104}$. Fix two such curves $C_a, C_b \in \mathcal{H}_1$ and let $\mathcal{A}$ be the set of curves in $\mathcal{F}(a,b)$ that intersects either C_a or C_b. By Lemma 4.1, there exists a subset $\mathcal{D}_1 \subset \mathcal{F}^d(a,b)$ such that each curve $C_i \in \mathcal{D}_1$ is disjoint to C_a, C_b, $\mathcal{A}$, and moreover

$$\chi(\mathcal{D}_1) \geq 2^{4\lambda_k+104} - 2^{\lambda_k+1} - k \cdot 2^{2\lambda_k+102} > 2^{4\lambda_k+103}.$$

Again by Lemma 3.1, there exists a subset $\mathcal{H}_2 \subset \mathcal{D}_1$ such that $\chi(\mathcal{H}_2) > 2^{\lambda_k}$, and for each pair of curves $C_u, C_v \in \mathcal{H}$ that intersect, $\chi(\mathcal{D}_1(u,v)) \geq 2^{3\lambda_k+102}$. Therefore, $\mathcal{H}_2$ contains k pairwise crossing curves $C_{i_1}, ..., C_{i_k}$ such that $i_1 < i_2 < \cdots < i_k$. Since $\chi(\mathcal{D}_1(i_1,i_2)) \geq 2^{3\lambda_k+102}$, by Lemma 4.4, there exists a subset $\mathcal{D}_2 \subset \mathcal{D}(i_1,i_2)$ such that every curve $C_l \in \mathcal{D}_2$ is disjoint to the set of curves $\{C_{i_1}, ..., C_{i_k}\}$ and

$$\chi(\mathcal{D}_2) \geq 2^{3\lambda_k+102} - 2^{2\lambda_k} > 2^{3\lambda_k+101}.$$

By applying Lemma 3.1 one last time, there exists a subset $\mathcal{H}_3 \subset \mathcal{D}_2$ such that $\chi(\mathcal{H}_3) > 2^{\lambda_k}$, and for every pair of curves $C_u, C_v \in \mathcal{H}_3$ that intersect, we have $\chi(\mathcal{D}_2(u,v)) \geq 2^{2\lambda_k+100}$. Therefore, $\mathcal{H}_3$ contains k pairwise intersecting curves $C_{j_1}, C_{j_2}, ..., C_{j_k}$ such that $i_1 < j_1 < j_2 < \cdots < j_k < i_2$. Since $\chi(\mathcal{D}_2(j_{k-1}, j_k)) \geq 2^{2\lambda_k+100}$, by Lemma 4.4, there exists a subset $\mathcal{D}_3 \subset \mathcal{D}_2(j_{k-1}, j_k)$ such that every curve $C_l \in \mathcal{D}_3$ is disjoint to the set of curves $\{C_{j_1}, ..., C_{j_k}\}$ (and disjoint to the set of curves $\{C_{i_1}, C_{i_2}, ..., C_{i_k}\}$) and

$$\chi(\mathcal{D}_3) \geq 2^{2\lambda_k+100} - 2^{2\lambda_k} > 2^{2\lambda_k+99}.$$

Now we can define a $2^{2\lambda_k+1}$-sequence $\{r_i\}_{i=0}^m$ of $\mathcal{D}_3$ such that $m \geq 4$. That is, we have subsets $\mathcal{D}_3[r_0, r_1], \mathcal{D}_3(r_1, r_2], ..., \mathcal{D}_3(r_{m-1}, r_m]$ that satisfies

1. $j_{k-1} < r_0 < r_1 < \cdots < r_m < j_k$, and
2. $\chi(\mathcal{D}_3[r_0, r_1]) = \chi(\mathcal{D}_3(r_1, r_2]) = \cdots = \chi(\mathcal{D}_3(r_{m-2}, r_{m-1}]) = 2^{2\lambda_k+1}$.

Fix a curve $C_q \in \mathcal{D}_3(r_1, r_2]$. See Figure 5(a).

Since $C_q \in \mathcal{D}_2(r_1, r_2] \subset \mathcal{F}^d$, there is a path $C_1, C_{p_1}, C_{p_2}, ..., C_{p_{d-1}}, C_q$ such that C_{p_t} is at distance t from C_1 for $1 \leq t \leq d-1$. Let R be the region enclosed by the y-axis, C_a and C_b. Since C_q lies inside of R, and C_1 lies outside of R, there must be a curve C_{p_t} that intersects either C_a or C_b for some $1 \leq t \leq d-1$. Since $C_a, C_b \in \mathcal{F}^d$ and $C_q \in \mathcal{D}_1$, $C_{p_{d-1}}$ must be this curve and we must have either $p_{d-1} < a$ or $p_{d-1} > b$. Now the proof splits into two cases.

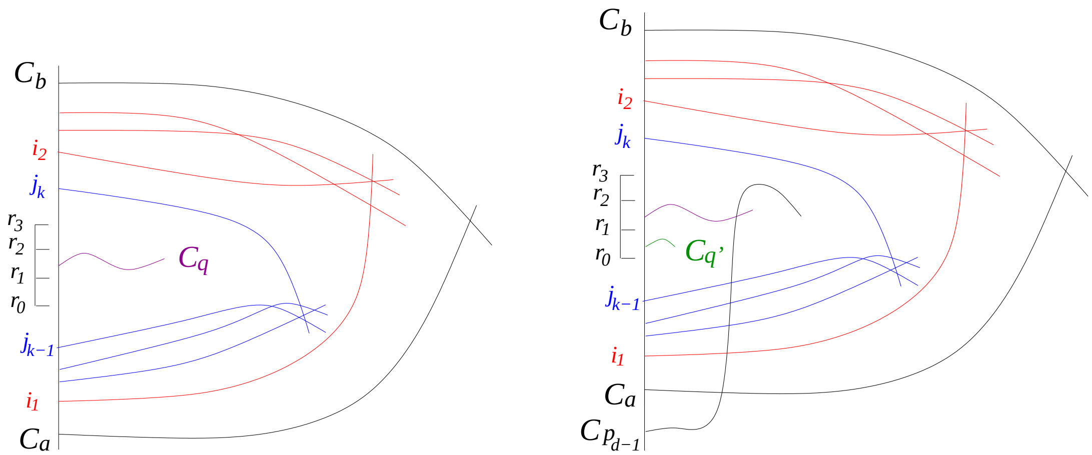
Case 1. Suppose $p_{d-1} < a$. Since

$$x(C_q) \leq x(\{C_{j_1}, C_{j_2}, ..., C_{j_{k-1}}\}),$$

the set of k curves $\mathcal{K} = \{C_{p_{d-1}}, C_{j_1}, C_{j_2}, ..., C_{j_{k-1}}\}$ are pairwise crossing. Now recall that $\chi(\mathcal{D}_3[r_0, r_1]) = 2^{2\lambda_k+1}$. By Lemma 4.4, there exists a curve $C_{q'} \in \mathcal{D}_3[r_0, r_1]$ such that the curve $C_{q'}$ does not intersect any of the members in $\{C_{p_{d-1}}, C_q\}$. See Figure 5(b). By construction of $\mathcal{D}_3[r_0, r_1]$, $C_{q'}$ does not intersect any of the curves in the set $\mathcal{K} \cup C_q$. By Lemma 4.3, we have

$$x(C_{q'}) \leq x(C_q) \leq x(\mathcal{K}).$$

Hence $\mathcal{K} \cup C_{q'}$ is a type 1 configuration.



(a) Curve C_q. (b) Curve $C_{q'}$.

Figure 5: Lemma 4.5.

Case 2. If $p_{d-1} > b$, then by a symmetric argument, $\mathcal{F}$ contains a type 2 configuration. $\square$

Lemma 4.6. *Let $\mathcal{F} = \{C_1, C_2, ..., C_n\}$ be a family of n x-monotone curves that form a left-flag. If $\chi(\mathcal{F}) > 2^{5\lambda_k+116}$, then $\mathcal{F}$ contains $k+1$ pairwise crossing members or a type 3 configuration.*

Proof. Assume that $\mathcal{F}$ does not contain $k+1$ pairwise crossing members. By Lemma 3.2, for some $d \geq 2$, the subset of curves $\mathcal{F}^d$ at distance d from C_1 satisfies

$$\chi(\mathcal{F}^d) \geq \frac{\chi(\mathcal{F})}{2} > 2^{5\lambda_k+115}.$$

Recall that for each curve $C_i \in \mathcal{F}^d$, there is a path $C_1, C_{p_1}, C_{p_2}, ..., C_{p_{d-1}}, C_i$ such that C_{p_t} is at distance t from C_1. Now we define subsets $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}^d$ as follows:

$$\mathcal{F}_1 = \{C_i \in \mathcal{F}^d : \text{there exists a path } C_1, C_{p_1}, C_{p_2}, ..., C_{p_{d-1}}, C_i \text{ with } p_{d-1} > i.\},$$

$$\mathcal{F}_2 = \{C_i \in \mathcal{F}^d : \text{there exists a path } C_1, C_{p_1}, C_{p_2}, ..., C_{p_{d-1}}, C_i \text{ with } p_{d-1} < i.\}.$$

Since $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}^d$, either $\chi(\mathcal{F}_1) \geq \chi(\mathcal{F}^d)/2$ or $\chi(\mathcal{F}_2) \geq \chi(\mathcal{F}^d)/2$. Since the following argument is the same for both cases, we will assume that

$$\chi(\mathcal{F}_1) \geq \frac{\chi(\mathcal{F}^d)}{2} \geq 2^{5\lambda_k+114}.$$

By Lemma 3.1, there exists a subset $\mathcal{H}_1 \subset \mathcal{F}_1$ such that $\chi(\mathcal{H}_1) > 2$, and for every pair of curves $C_a, C_b \in \mathcal{H}_1$ that intersect, $\mathcal{F}_1(a,b) \geq 2^{5\lambda_k+112}$. Fix two such curves $C_a, C_b \in \mathcal{H}_1$ and let $\mathcal{A}$ be the set of curves in $\mathcal{F}(a,b)$ that intersects either C_a or C_b. By Lemma 4.1, there exists a subset $\mathcal{D}_1 \subset \mathcal{F}_1(a,b)$ such that each curve $C_i \in \mathcal{D}_1$ is disjoint to C_a, C_b, $\mathcal{A}$, and moreover

$$\chi(\mathcal{D}_1) \geq 2^{5\lambda_k+112} - 2^{\lambda_k+1} - k \cdot 2^{2\lambda_k+102} > 2^{5\lambda_k+111}.$$

Again by Lemma 3.1, there exists a subset $\mathcal{H}_2 \subset \mathcal{D}_1$ such that $\chi(\mathcal{H}_2) > 2^{\lambda_k}$, and for every pair of curves $C_u, C_v \in \mathcal{H}_2$ that intersect, $\chi(\mathcal{D}_1(u,v)) \geq 2^{4\lambda_k+110}$. Therefore, $\mathcal{H}_2$ contains k pairwise crossing members $C_{i_1}, C_{i_2} ..., C_{i_k}$ for $i_1 < i_2 < \cdots < i_k$. Since $\chi(\mathcal{D}_1(i_1,i_2)) \geq 2^{4\lambda_k+110}$, by Lemma 4.4, there exists a subset $\mathcal{D}_2 \subset \mathcal{D}_1(i_1,i_2)$ such that

$$\chi(\mathcal{D}_2) > 2^{4\lambda_k+110} - 2^{2\lambda_k} > 2^{4\lambda_k+109},$$

and each curve $C_l \in \mathcal{D}_2$ is disjoint to the set of curves $\{C_{i_1}, C_{i_2}, ..., C_{i_k}\}$. Now we define a $2^{4\lambda_k+107}$-sequence $\{r_i\}_{i=0}^m$ of $\mathcal{D}_2$ such that $m \geq 4$. Therefore, we have subsets

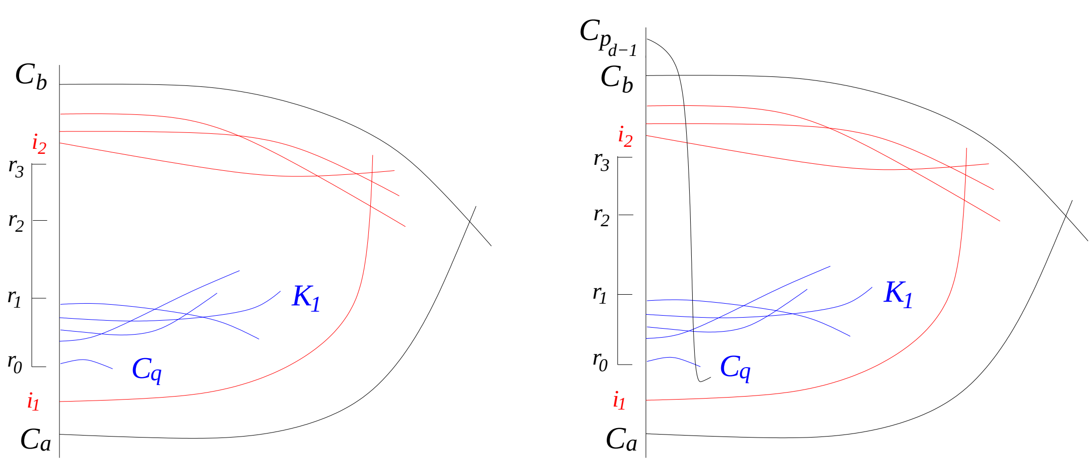
$$\mathcal{D}_2[r_0,r_1], \mathcal{D}_2(r_1,r_2], ..., \mathcal{D}_2(r_{m-1},r_m]$$

such that

$$\chi(\mathcal{D}_2[r_0,r_1]) = \chi(\mathcal{D}_2(r_1,r_2]) = 2^{4\lambda_k+107}.$$

By Lemma 4.5, we know that $\mathcal{D}_2[r_0,r_1]$ contains either a type 1 or type 2 configuration.

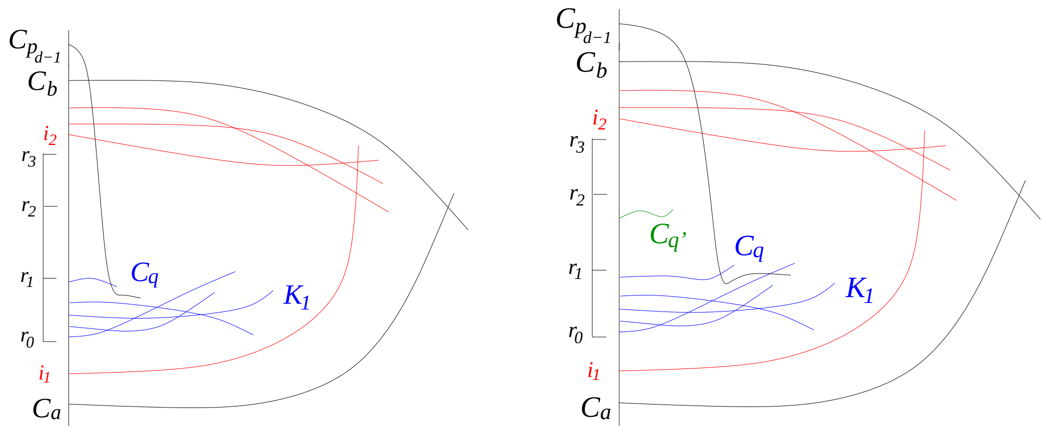
Suppose that $\mathcal{D}_2[r_0,r_1]$ contains a type 2 configuration $\{\mathcal{K}_1, C_q\}$, where $\mathcal{K}_1$ is the set of k pairwise intersecting curves. See Figure 6(a). $C_q \in \mathcal{D}_2[r_0,r_1] \subset \mathcal{F}^d$ implies that there exists a path $C_1, C_{p_1}, C_{p_2}, ..., C_{p_{d-1}}, C_q$ such that $p_{d-1} > q$. Let R be the region enclosed by the y-axis, C_a and C_b. Since C_q lies inside of R, and C_1 lies outside of R, there must be a curve C_{p_t} that intersects either C_a or C_b for some $1 \leq t \leq d-1$. Since $C_a, C_b \in \mathcal{F}^d$, $C_{p_{d-1}}$ must be this curve. Moreover, $C_q \in \mathcal{D}_1 \subset \mathcal{F}_1$ implies that $p_{d-1} > b$. Since our curves are x-monotone and $x(C_q) \leq x(\mathcal{K}_1)$, $C_{p_{d-1}}$ intersects all of the curves in $\mathcal{K}_1$. This creates $k+1$ pairwise crossing members in $\mathcal{F}$ and we have a contradiction. See Figure 6(b).



(a) Type 2 configuration. (b) $k+1$ pairwise crossing curves.

Figure 6: Lemma 4.6.

Therefore we can assume that $\mathcal{D}_2[r_0,r_1]$ contains a type 1 configuration $\{\mathcal{K}_1, C_q\}$ where $\mathcal{K}_1$ is the set of k pairwise intersecting curves. See Figure 7(a). By the same argument as above, there exists a curve $C_{p_{d-1}}$ that intersects C_q such that $p_{d-1} > b$. Hence $\mathcal{K}_2 = \{C_{i_2}, C_{i_3}, ..., C_{i_k}, C_{p_{d-1}}\}$ is a set of k pairwise intersecting curves. Since $\chi(D_2(r_1,r_2]) = 2^{4\lambda_k+107}$, Lemma 4.4 implies that there exists a curve $C_{q'} \in \mathcal{D}_1(r_1,r_2]$ that does not intersect any members in the set $\{C_q, C_{p_{d-1}}\}$.



(a) Type 1 configuration.

(b) Type 3 configuration.

Figure 7: Lemma 4.6.

By construction of $\mathcal{D}_2(r_1, r_2]$ and by the definition of a type 1 configuration, $C_{q'}$ does not intersect any members in the set $\{\mathcal{K}_1, \mathcal{K}_2, C_q\}$. By Lemma 4.3, we have

$$x(C_{q'}) \leq x(C_q) \leq x(\mathcal{K}_1 \cup \mathcal{K}_2),$$

and therefore $\mathcal{K}_1, \mathcal{K}_2, C_{q'}$ is a type 3 configuration. See Figure 7(b).

$\square$

5 Proof of the Theorem 2.1

The proof is by induction on k. The base case $k = 2$ is trivial. Now suppose that the statement is true up to k. Let $\mathcal{F} = \{C_1, C_2, ..., C_n\}$ be a collection of n simple x-monotone curves that form a left-flag, such that $\chi(\mathcal{F}) > 2^{(5^{k+2}-121)/4}$. We will show that $\mathcal{F}$ contains $k+1$ pairwise crossing members. We define the recursive function λ_k such that $\lambda_2 = 1$ and

$$\lambda_k = 5\lambda_{k-1} + 121 \qquad \text{for} \qquad k \geq 3.$$

This implies that $\lambda_k = (5^{k+1} - 121)/4$ for all $k \geq 2$. Therefore, we have

$$\chi(\mathcal{F}) > 2^{(5^{k+2}-121)/4} = 2^{\lambda_{k+1}} = 2^{5\lambda_k + 121}.$$

Just as before, there exists an integer $d \geq 2$, such that the set of curves $\mathcal{F}^d \subset \mathcal{F}$ at distance d from the curve C_1 satisfies

$$\chi(\mathcal{F}^d) \geq \frac{\chi(\mathcal{F})}{2} > 2^{5\lambda_k + 120}.$$

Now we can assume that $\mathcal{F}^d$ does not contain $k+1$ pairwise crossing members, since otherwise we would be done. By Lemma 3.1, there exists a subset $\mathcal{H} \subset \mathcal{F}^d$ such that $\chi(\mathcal{H}) > 2$, and for every pair of curves $C_a, C_b \in \mathcal{H}$ that intersect, $\mathcal{F}^d(a,b) \geq 2^{5\lambda_k + 118}$. Fix two such curves $C_a, C_b \in \mathcal{H}$, and let $\mathcal{A}$ be the set of curves in $\mathcal{F}(a,b)$ that intersects C_a or C_b. By Lemma 4.1, there exists a subset $\mathcal{D} \subset \mathcal{F}^d(a,b)$ such that each curve $C_i \in \mathcal{D}$ is disjoint to C_a, C_b, $\mathcal{A}$, and moreover

$$\chi(\mathcal{D}) \geq 2^{5\lambda_k+118} - 2^{\lambda_k+1} - 2^{2\lambda_k+102} \geq 2^{5\lambda_k+117}.$$

By Lemma 4.6, $\mathcal{D}$ contains a type 3 configuration $\{\mathcal{K}_1, \mathcal{K}_2, C_q\}$, where $\mathcal{K}_t$ is a set of k pairwise intersecting curves for $t \in \{1,2\}$. See Figure 8. Just as argued before, since $C_q \in \mathcal{D} \subset \mathcal{F}^d$, there exists a path $C_1, C_{p_1}, C_{p_2}, ..., C_{p_{d-1}}, C_q$ in $\mathcal{F}$ such that either $p_{d-1} < a$ or $p_{d-1} > b$. Since our curves are x-monotone and $x(C_q) \leq x(\mathcal{K}_1 \cup \mathcal{K}_2)$, this implies that either $\mathcal{K}_1 \cup C_{p_{d-1}}$ or $\mathcal{K}_2 \cup C_{p_{d-1}}$ are $k+1$ pairwise crossing curves. See Figures 9(a) and 9(b).

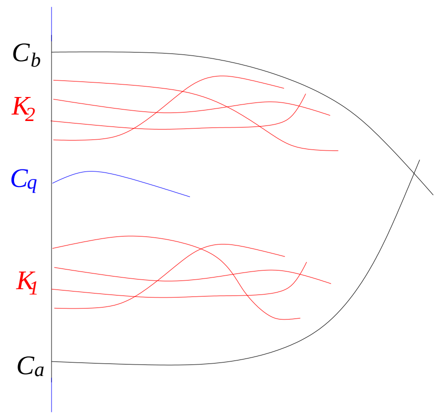
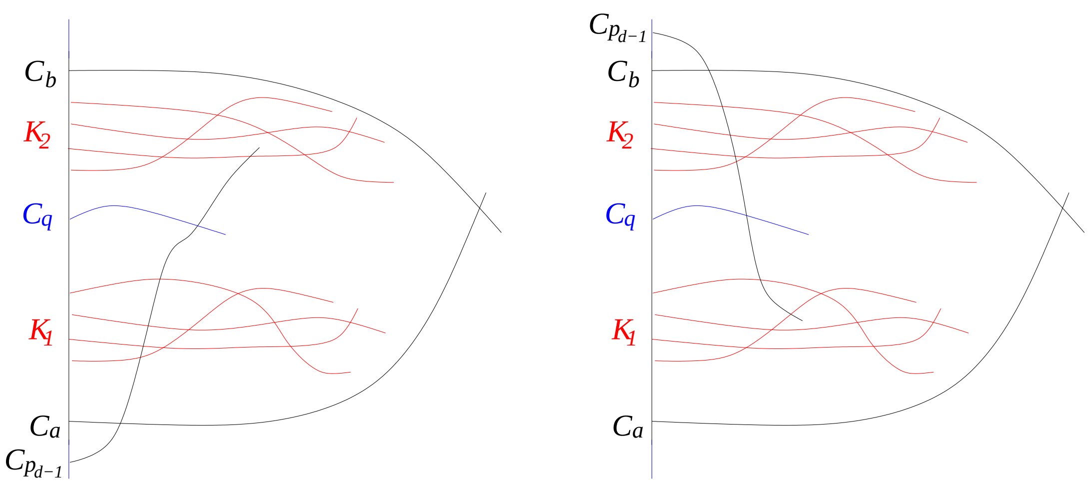


Figure 8: Type 3 configuration.



(a) $\mathcal{K}_1 \cup C_{p_{d-1}}$ are $k+1$ pairwise crossing curves.

(b) $\mathcal{K}_2 \cup C_{p_{d-1}}$ are $k+1$ pairwise crossing curves.

Figure 9: $k+1$ pairwise crossing curves.

$\square$

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